Mathematical Optimization Models for the Euclidean Steiner Tree Problem in R^d

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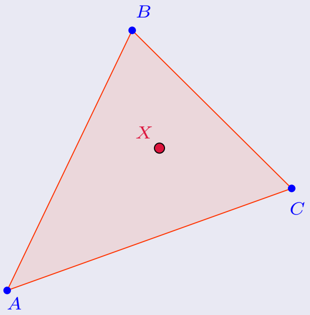
Challenge of Fermat in the 17th century

Given three points in the plane, find a fourth point such that the sum of its distance to the three given points is minimum.

Triangle: Three given points



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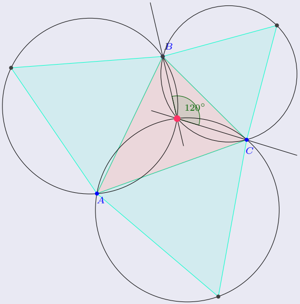

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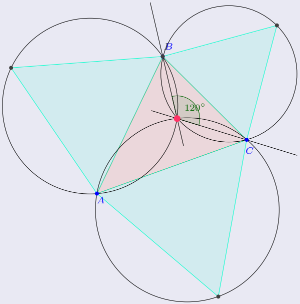

- **Torricelli** (1647) pointed out a solution when the triangle formed by the three given points does not have an angle ≥ 120.

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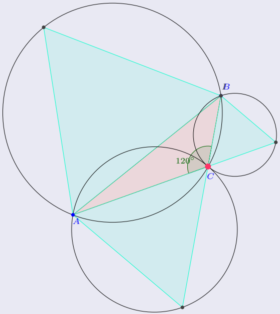

- **Torricelli** (1647) pointed out a solution when the triangle formed by the three given points does not have an angle ≥ 120.
- **Heinen** (1837) apparently is the first to prove that, for a triangle in which an angle is ≥ 120, the vertex associated with this angle is the minimizing point.

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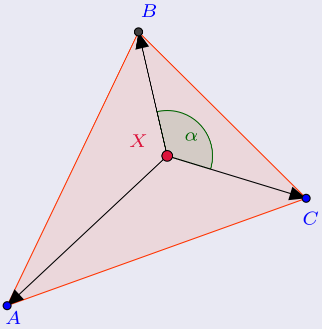
Fermat's Challenge as an Optimization Problem



Minimize $\mathcal{D} = ||\overrightarrow{XA}|| + ||\overrightarrow{XB}|| + ||\overrightarrow{XC}||$

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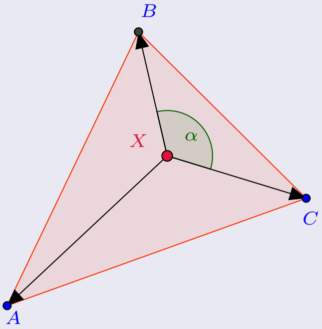

Minimize $\mathcal{D} = ||\overrightarrow{XA}|| + ||\overrightarrow{XB}|| + ||\overrightarrow{XC}||$

The solution is given when

$$\nabla\mathcal{D} = 0.$$

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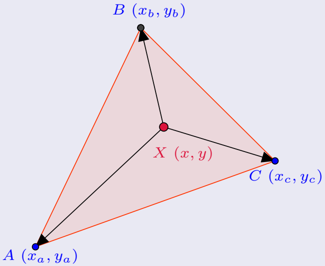

$$\begin{aligned} ||\overrightarrow{XA}|| &= \sqrt{(x_a - x)^2 + (y_a - y)^2} \\ ||\overrightarrow{XB}|| &= \sqrt{(x_b - x)^2 + (y_b - y)^2} \\ ||\overrightarrow{XC}|| &= \sqrt{(x_c - x)^2 + (y_c - y)^2} \end{aligned}$$

$$\nabla \mathcal{D} = \begin{pmatrix} \frac{\partial \mathcal{D}}{\partial x} \\ \frac{\partial \mathcal{D}}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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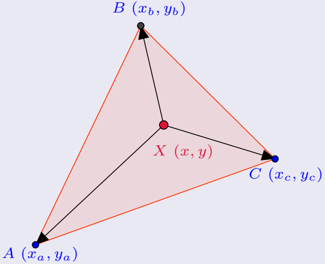

$$\frac{\partial \mathcal{D}}{\partial x} = \frac{x_a - x}{||\overrightarrow{XA}||} + \frac{x_b - x}{||\overrightarrow{XB}||} + \frac{x_c - x}{||\overrightarrow{XC}||} = 0$$

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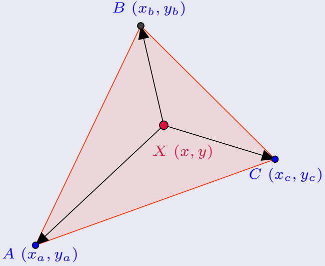

$$\begin{pmatrix} \dfrac{\partial \mathcal{D}}{\partial x} \\ \\ \dfrac{\partial \mathcal{D}}{\partial y} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{x_a - x}{||\overrightarrow{XA}||} \\ \\ \dfrac{y_a - y}{||\overrightarrow{XA}||} \end{pmatrix} + \begin{pmatrix} \dfrac{x_b - x}{||\overrightarrow{XB}||} \\ \\ \dfrac{y_b - y}{||\overrightarrow{XB}||} \end{pmatrix} + \begin{pmatrix} \dfrac{x_c - x}{||\overrightarrow{XC}||} \\ \\ \dfrac{y_c - y}{||\overrightarrow{XC}||} \end{pmatrix}}_{\text{Unitary Vectors Sum}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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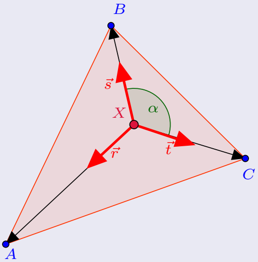

Three Forces in Equilibrium

$$\nabla \mathcal{D} = \vec{r} + \vec{s} + \vec{t} = \vec{0}$$

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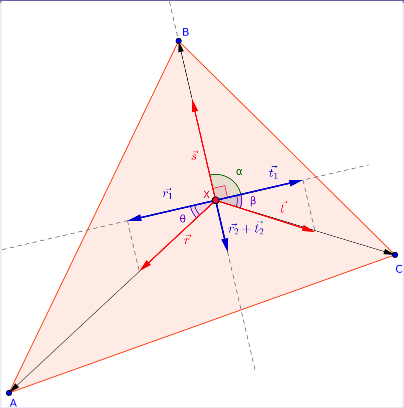

Three Forces in Equilibrium
$(0° < \theta, \beta < 90°)$

$$||\vec{r_1}|| = ||\vec{t_1}|| \Rightarrow \cos(\theta) = \cos(\beta) \Rightarrow \theta = \beta$$

$$||\vec{r_2} + \vec{t_2}|| = ||\vec{s}|| \Rightarrow \sin(\theta) + \sin(\beta) = 1 \Rightarrow \sin(\theta) = \sin(\beta) = \frac{1}{2} \Rightarrow \theta = \beta = 30°$$

$$\alpha = 90° + \beta \Rightarrow \alpha = 120°.$$

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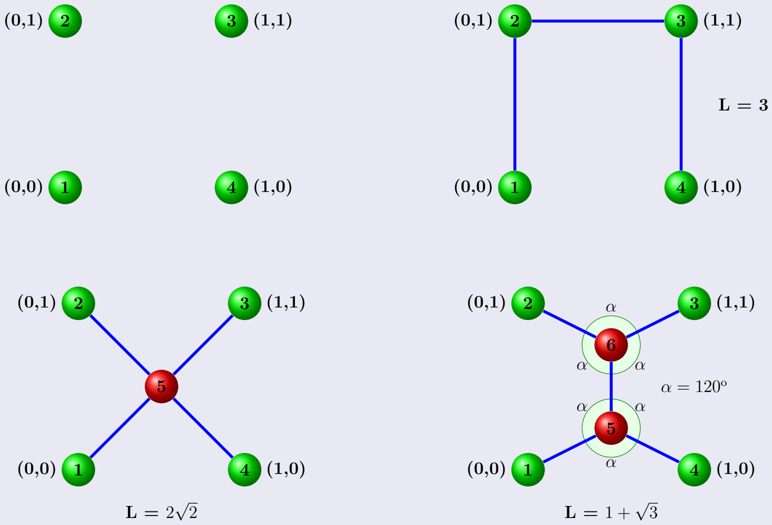
An example with four points in the plane...



Now, consider p given points in $\mathbb{R}^n$.

Steiner Minimal Tree Problem

Find a minimum tree that spans these points using or not extra points, which are called Steiner points.

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Steiner Minimal Tree Problem

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- This problem has been shown to be NP-Hard.
- All distances are considered to be Euclidean.

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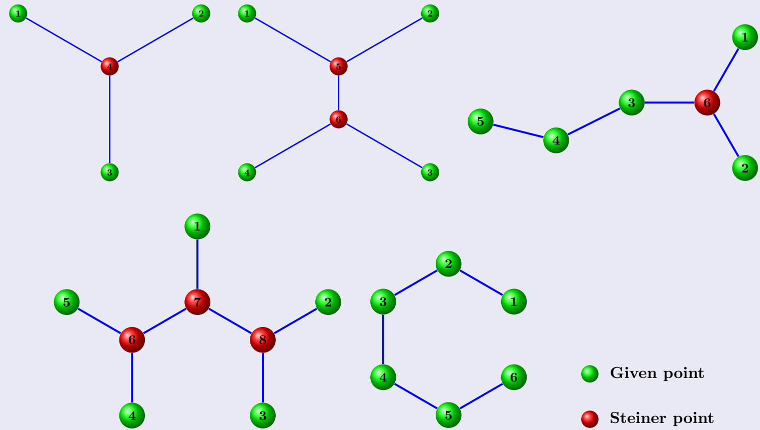
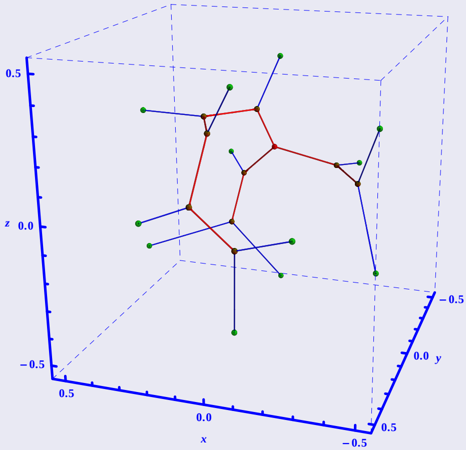
Problem Definition

Some examples of Steiner points in $\mathbb{R}^2$



An example in $\mathbb{R}^3$: Icosahedron



Number of Steiner Points

Given p points $x^i \in \mathbb{R}^n$, $i = 1, 2, \ldots, p$, the *maximum number of Steiner points* is $p - 2$.

Properties

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Degree of Steiner Points

A nondegenerated Steiner point has degree (valence) *equal to 3*.

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Steiner Points Edges

The edges emanating from a nondegenerated Steiner point *lie in a plane* and have mutual angle *equal to 120°*.

Steiner Topology

It is a topology that satisfies all the Steiner Tree properties.

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Number of Topologies (Gilbert and Pollack)

The total number of different topologies with k Steiner points is

$$C_{p,k+2}\frac{(p+k-2)!}{k!2^k},$$

where p is the number of given points in $\mathbb{R}^n$.

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Full Steiner Topologies ($k = p - 2$)

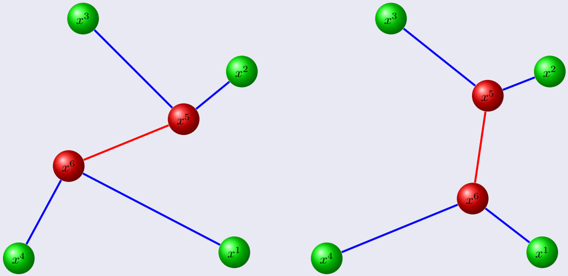
The total number of different topologies with $k = p - 2$ Steiner points is

$$1 \cdot 3 \cdot 5 \cdot 7 \ldots (2p-5) = (2p-5)!!.$$

For example, if $p = 10$, the Number of Full Steiner Topologies is equal to

$$15!! = 2,027,025.$$

Example of Local Optimization



Finding the best solution...

$$\text{Minimize } ||x^3 - x^5|| + ||x^2 - x^5|| + ||x^5 - x^6|| + ||x^1 - x^6|| + ||x^4 - x^6||$$

subject to x^5 and $x^6 \in \mathbb{R}^n$.

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Given p different points in R^n, the ESTP seeks to find a minimum tree that spans these points using or not extra points, which are called Steiner points. The length of each edge is the Euclidean distance between its ends.

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We consider a special graph $G = (V, E)$ as follows:
Let $P = \{1, 2, ..., p-1, p\}$ be the set of indices associated with the given points in R^n : $x^1, x^2, ..., x^{p-1}, x^p$, and a set of indices $S = \{p+1, p+2, ..., 2p-3, 2p-2\}$ associated with the Steiner points also in R^n : $x^{p+1}, x^{p+2}, ..., x^{2p-3}, x^{2p-2}$.
We take $V = P \cup S$. We denote $[i, j]$ $i < j$, i and $j \in V$ an edge of G.
Thus we define $E_1 = \{[i, j] \mid i \in P, \ j \in S\}$, $E_2 = \{[i, j] \mid i < j, \ i \ and \ j \in S\}$, and $E = E_1 \cup E_2$.
A tree which is an optimal solution for the ESTP is a sub-graph of $G = (V, E)$.
We consider the following variables:

$$x^i \in \mathbb{R}^n, \ i \in S, \tag{1}$$

$$y_{ij} \in \{0, 1\}, \quad [i, j] \in E. \tag{2}$$

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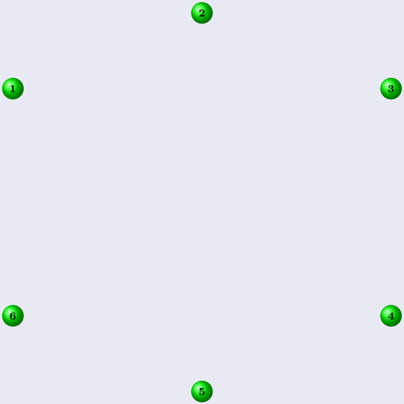
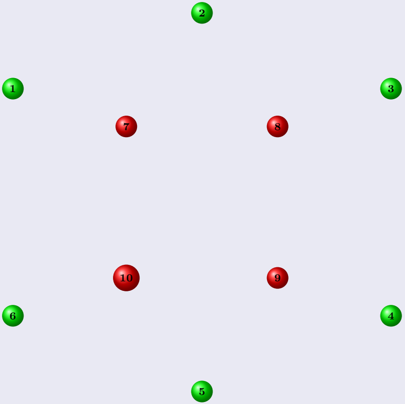
First Formulation: an example with $p = 6$

- 6 given points.



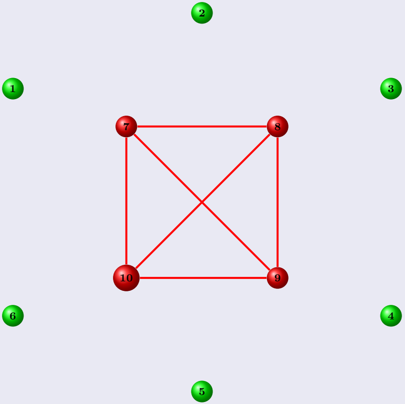
First Formulation: an example with $p = 6$

- 6 given points.
- 4 Steiner points.



First Formulation: an example with $p = 6$

- 6 given points.
- 4 Steiner points.
- All possible edges among Steiner points.



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First Formulation: an example with $p = 6$

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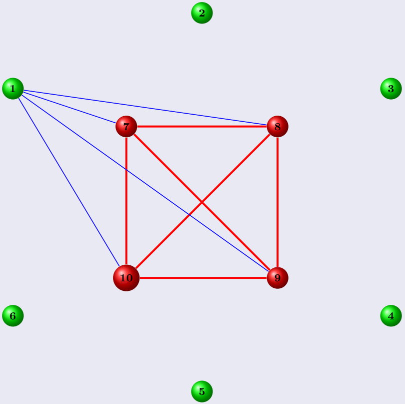
- All possible edges among Steiner points.
- All possible connections between a given point and a Steiner point.



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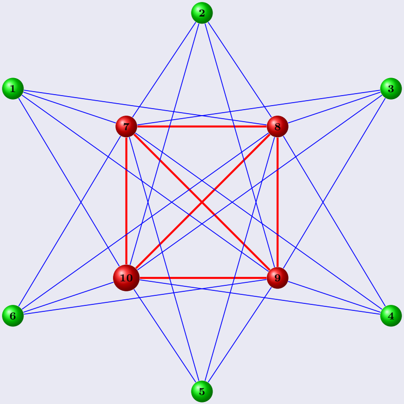
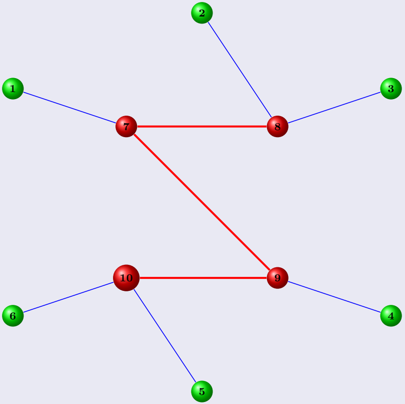


First Formulation: an example with $p = 6$

- 6 given points.
- 4 Steiner points.
- All possible edges among Steiner points.
- All possible connections between a given point and a Steiner point.
- All possible edges.
- An example of a set of possible edges.



Maculan-Michelon-Xavier (2000) [1]

$$(\mathrm{P1}): \quad \text{Minimize} \sum_{[\mathbf{i},\mathbf{j}]\in \mathbf{E}} ||\mathrm{x}^{\mathbf{i}} - \mathrm{x}^{\mathbf{j}}||\, \mathrm{y}_{\mathbf{ij}} \text{ subject to} \tag{3}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P, \tag{4}$$

$$\sum_{i<j, i\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{5}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S, \tag{6}$$

$$x^i \in \mathbb{R}^n, \quad i \in S, \tag{7}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E, \tag{8}$$

We consider $||x^i - x^j|| \approx \sqrt{\sum_{k=1}^{n}(x_k^i - x_k^j)^2 + \lambda^2}$

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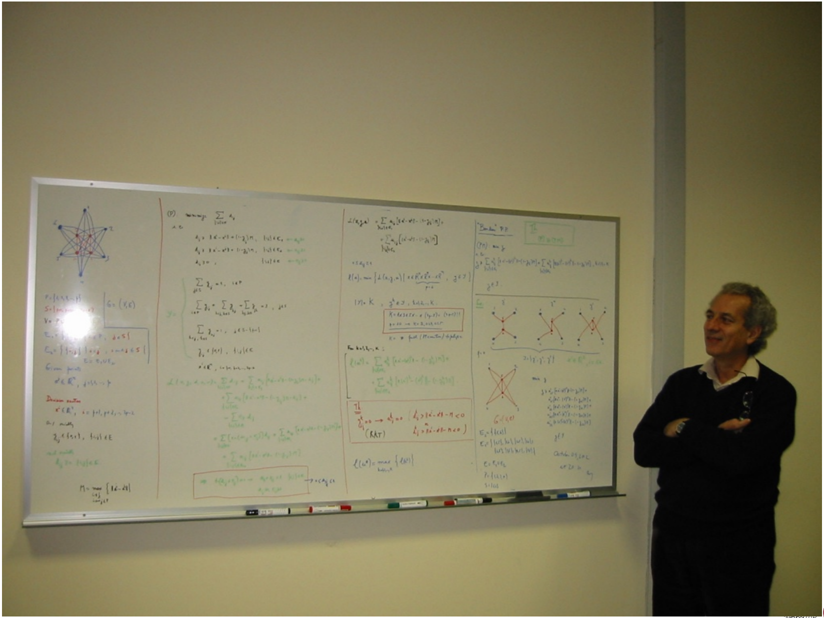
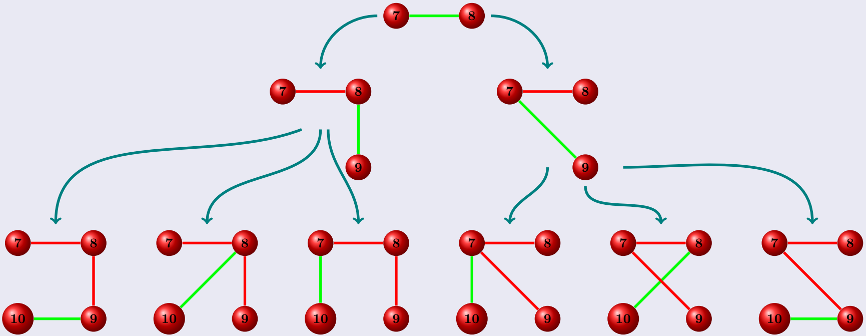


First Formulation: an example with $p = 6$

$$\sum_{k<j, k \in S} y_{kj} = 1, \; j \in S - \{p+1\}$$

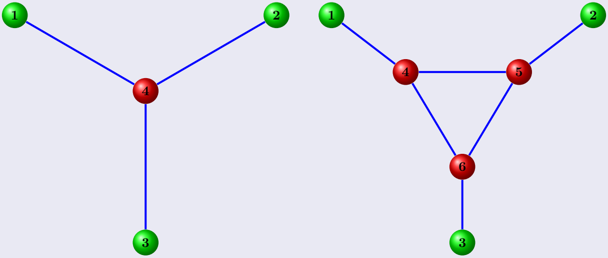
$$\begin{aligned} y_{7,8} &= 1 \\ y_{7,9} + y_{8,9} &= 1 \\ y_{7,10} + y_{8,10} + y_{9,10} &= 1 \end{aligned}$$



First Formulation: another example

If we don't considerer

$$\sum_{k<j, k \in S} y_{kj} = 1, \quad j \in S - \{p+1\}$$



Fampa-Maculan (2001,2004) [2,3]

$$(P2): \quad \text{Minimize} \sum_{[\mathbf{i},\mathbf{j}]\in \mathbf{E}} \mathrm{d}_{\mathbf{ij}} \text{ subject to} \tag{9}$$

$$d_{ij} \geqslant ||x^i - x^j|| - M(1 - y_{ij}), \; [i,j] \in E, \tag{10}$$

$$d_{ij} \geqslant 0, \; [i,j] \in E \tag{11}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P, \tag{12}$$

$$\sum_{i<j, i\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{13}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S, \tag{14}$$

$$x^i \in \mathbb{R}^n, \; i \in S, \tag{15}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E, \tag{16}$$

$$d_{ij} \in \mathbb{R}. \tag{17}$$

We consider $\begin{cases} ||x^i - x^j|| \approx \sqrt{\sum_{k=1}^{n} (x^i_k - x^j_k)^2 + \lambda^2} \\ M = maximum\{||x^i - x^j|| \text{ for } 1 \leqslant i \leqslant j \leqslant p\} \text{ in general,} \end{cases}$

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MINLP: Formulations for the Euclidean Steiner Problem

Second Formulation (First Property)

If $\bar{x}^j \in R^n,\ j \in S$ and $\bar{y}_{ij} \in \{0,1\},\ [i,j] \in E$ is an optimal solution, then

- $d_{ij} = ||a^i - \bar{x}^j|| \geqslant 0$ or $d_{ij} = 0$, for all $[i,j] \in E_1$ and
- $d_{ij} = ||\bar{x}^i - \bar{x}^j|| \geqslant 0$ or $d_{ij} = 0$, for all $[i,j] \in E_2$.

Second Formulation (Second Property)

$y_{ij} \in \{0,1\},\ [i,j] \in E$ is associated with a full Steiner Topology if, and only if, the following equations are satisfied:

$$\begin{aligned} \sum_{j \in S} y_{ij} &= 1, \quad i \in P, \\ \sum_{k<j, k \in S} y_{kj} &= 1, \quad j \in S - \{p+1\}, \\ \sum_{i \in P} y_{ij} + \sum_{k<j, k \in S} y_{kj} + \sum_{k>j, k \in S} y_{jk} &= 3, \quad j \in S, \end{aligned}$$

Second Formulation (Third Property)

In a minimum Steiner tree with more than three terminal nodes, all Steiner points have no more than two connections with terminal nodes. So, if $p > 3$,

$$\sum_{i \in P} y_{ij} \leqslant 2,\ j \in S.$$

Note that...

When we consider

$$||x^i - x^j|| \approx \sqrt{\sum_{l=1}^{n}(x^i_l - x^j_l)^2 + \lambda^2},$$

error propagations may happen.

MINLP: Formulations for the Euclidean Steiner Problem

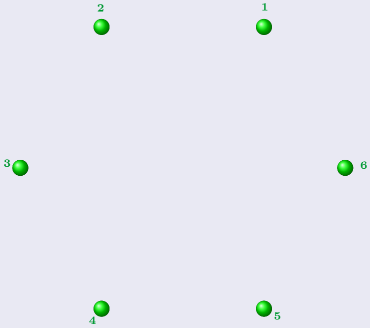
Note that...

When we consider

$$||x^i - x^j|| \approx \sqrt{\sum_{l=1}^{n}(x_l^i - x_l^j)^2 + \lambda^2},$$

error propagations may happen.

Example: Regular Hexagon



- 6 given points.
- Each given point is in a vertex of a Regular Hexagon.
- Each side of the Hexagon is equal to 1.

Note that...

When we consider

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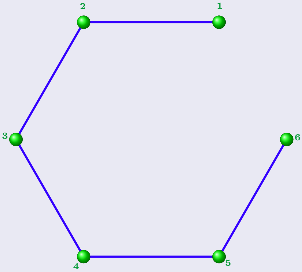
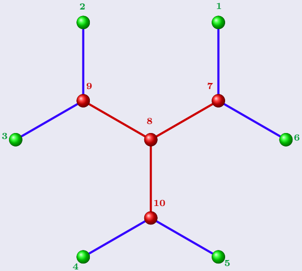
error propagations may happen.

Example: Regular Hexagon





- Objective Function: 5
- $\lambda^2 = 10^{-8}$

- Objective Function: $5.196 = 3\sqrt{3}$
- $\lambda^2 = 10^{-6}$

Ouzia-Maculan (2018) [4]

$$(P3): \quad \text{Minimize} \sum_{[i,j]\in E} \sqrt{\sum_{k=1}^{n} d_{ijk}^2} \quad \text{subject to} \tag{18}$$

$$-y_{ij} \leq d_{ijk} \leq y_{ij}, \quad [i,j] \in E,\ k = 1,2,\ldots,n, \tag{19}$$

$$-(1-y_{ij}) + (x_k^i - x_k^j) \leq d_{ijk} \leq (x_k^i - x_k^j) + (1-y_{ij}), \quad [i,j] \in E,\ k = 1,2,\ldots,n, \tag{20}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P, \tag{21}$$

$$\sum_{i<j, i\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{22}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S, \tag{23}$$

$$x^i \in \mathbb{R}^n, \quad i \in S, \tag{24}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E, \tag{25}$$

$$d_{ijk} \in \mathbb{R}. \tag{26}$$

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Ouzia-Maculan (2018) [4]

$$(\text{P4}): \quad \text{Minimize} \sum_{[\mathbf{i},\mathbf{j}]\in \mathbf{E}} \sqrt{\mathrm{d}_{\mathbf{ij}}} \quad \text{subject to} \tag{27}$$

$$d_{ij} \geqslant \sum_{k=1}^{n}(x_k^i - x_k^j)^2 - (1 - y_{ij}),\ [i,j] \in E, \tag{28}$$

$$d_{ij} \geqslant 0,\ [i,j] \in E \tag{29}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P, \tag{30}$$

$$\sum_{i<j, i\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{31}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S, \tag{32}$$

$$x^i \in \mathbb{R}^n,\ i \in S, \tag{33}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E, \tag{34}$$

$$d_{ij} \in \mathbb{R}. \tag{35}$$

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Maculan-Ouzia-Pinto (2020) [5]

$$(P5): \quad \text{Minimize} \sum_{[\mathbf{i},\mathbf{j}]\in \mathbf{E}} \mathrm{d}_{\mathbf{ij}} \quad \text{subject to} \tag{36}$$

$$d_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2, \quad [i,j] \in E, \tag{37}$$

$$-y_{ij} \leq t_{ijk} \leq y_{ij}, \quad [i,j] \in E, \ k = 1,2,\ldots,n, \tag{38}$$

$$-(1-y_{ij}) + (x_k^i - x_k^j) \leq t_{ijk} \leq (x_k^i - x_k^j) + (1-y_{ij}), \quad [i,j] \in E, \ k = 1,2,\ldots,n, \tag{39}$$

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P, \tag{40}$$

$$\sum_{i<j, i\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{41}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S, \tag{42}$$

$$x^i \in \mathbb{R}^n, \ i \in S, \tag{43}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E, \tag{44}$$

$$d_{ij} \geq 0, \quad [i,j] \in E. \tag{45}$$

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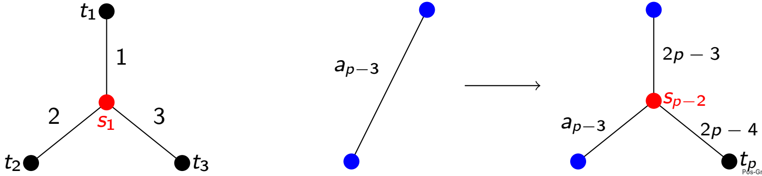
Maculan-Ouzia-Pinto (2020) [5]

We added to model (P5) many families of constraints to eliminate isomorphic trees, based on the idea presented by Smith in his paper [6].

The author presents a bijection between every possible full Steiner topology on p terminal points and vectors in $a \in \mathbb{N}^{p3}$, where $1 \leq a_i \leq 2i + 1, \quad i = 1, ..., p3$.

The edges are labeled and each a_i indicates which edge the $(i + 1)$-th Steiner point will be placed on.

We call the resulting model (P6).



To represent the Smith vector, we define the binary variables

$$v_{ij} \in \{0,1\}, \qquad i = 1,\ldots,p-3, \quad j = 1,\ldots,2i+1,$$

that assume value 1 if $a_i = j$. We must have

$$\sum_{j=1}^{2i+1} v_{ij} = 1, \quad \forall\, i = 1,\ldots,p-3. \tag{46}$$

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To control the edges labeling, we define the binary variables

$$e_{ijkl} \in \{0,1\}, \qquad i = 0,\ldots,p-3,\ j = 1,\ldots,2i+3,\ k \in \{1,2\},\ l = 1,\ldots,2p-2.$$

We will have $e_{ijkl} = 1$ if, in stage i of the construction of the tree, vertex k of edge j is l.

We must have

$$\sum_{l=1}^{2p-2} e_{ijkl} = 1, \quad \forall\, i,j,k. \tag{47}$$

along with some initial conditions (corresponding to the null-vector or the 3-terminal topology):

$$e_{0j1j} = 1, \qquad j = 1,2,3, \tag{48}$$

$$e_{0j2(p+1)} = 1, \qquad j = 1,2,3. \tag{49}$$

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In iteration $i > 0$, the new Steiner point will be placed in the middle of edge a_i, or, in our notation, in the middle of edge j, such that $v_{ij} = 1$. We write, for all $i = 1, \ldots, p-3,\ j = 1, \ldots, 2i+3,\ l = 1, \ldots, 2p-2$,

$$e_{ij1l} = e_{(i-1)j1l}, \tag{50}$$

$$-v_{ij} + e_{(i-1)j2l} \leq e_{ij2l} \leq e_{(i-1)j2l} + v_{ij}, \tag{51}$$

$$v_{ij} \leq e_{ij2(i+1+p)} \leq 2 - v_{ij}. \tag{52}$$

For the two new edges added at each iteration, we have, for all $i = 1, \ldots, p-3,\ l = 1, \ldots, 2p-2$,

$$e_{i(2i+2)1(i+3)} = 1, \tag{53}$$

$$e_{i(2i+2)2(i+1+p)} = 1, \tag{54}$$

$$e_{i(2i+3)1l} = \sum_{j=1}^{2i+1} v_{ij} \cdot e_{(i-1)j2l}, \tag{55}$$

$$e_{i(2i+3)2(i+1+p)} = 1. \tag{56}$$

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We can linearize the binary product in equation (55) using the McCormick inequalities.

Finally, we relate variables e_{ijkl} to the variables y_{ij}. For all
$i = 1, \dots, 2p-2,\ j = p+1, \dots, 2p-2,\ i < j,\ k = 1, \dots, 2p-3,$

$$y_{ij} \geq e_{(p-3)k1i} + e_{(p-3)k2j} - 1. \tag{57}$$

The experiments were performed on a machine equipped with an Intel(R) Xeon(R) i7 8700 CPU @ 3.20GHz with 12 cores and 64GB DRAM of memory.

Models (P1) to (P4) were solved using BARON 21.1.7.
Models (P5) and (P6) were solved using XPRESS 8.11.0.

	Tetrahedron		
	obj	time (s)	nodes
P1	0.813	9	839
P2	0.813	0.5	11
P3	0.955	0.09	5
P4	0.813	300	60493
P5	0.813	0.01	11
P6	0.813	0.01	5

	Octahedron			
	obj	time (s)	nodes	gap
P1	0.9560	10800*	160633	95.9
P2	0.9560	277	1803	0
P3	1.2075	38	1517	0
P4	0.9562	10800*	418903	82.7
P5	0.9560	1	1797	0
P6	0.9560	0.01	209	0

* = Execution aborted after 3 hours.

	Cube			
	obj	time (s)	nodes	gap
P1	1.1924	10800*	79513	99.9
P2	1.1924	10800*	30291	99.9
P3	1.3458	10800*	128019	79.8
P4	1.2003	10800*	151437	99.9
P5	1.1924	634	1043859	0
P6	1.1924	16	20869	0

* = Execution aborted after 3 hours.

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Numerical experiments

	Icosahedron			
	obj	time (s)	nodes	gap
P1	1.6571	10800*	877	99.9
P2	1.6649	10800*	1025	99.9
P3	1.98	10800*	1616	99.8
P4	1.7953	10800*	4253	99.9
P5	1.6319	10800*	11153041	94.2
P6	1.6256	10800*	5192564	89.2

* = Execution aborted after 3 hours.

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	Dodecahedron			
	obj	time (s)	nodes	gap
P1	2.2617	10800*	936	99.9
P2	2.5618	10800*	14	99.9
P3	3.182	10800*	222	99.7
P4	2.602	10800*	38	99.9
P5	2.3516	10800*	3888898	99.9
P6	2.6449	10800*	98170	99.9

* = Execution aborted after 3 hours.

Main idea: starting with the Fermat point, add one Steiner point at a time.

Before all Steiner points are added, there will be at least one Steiner point with degree greater than 3. Let s be one of these points. A new Steiner point will be added to the tree to reduce the degree of s.

Heuristic - Example 1

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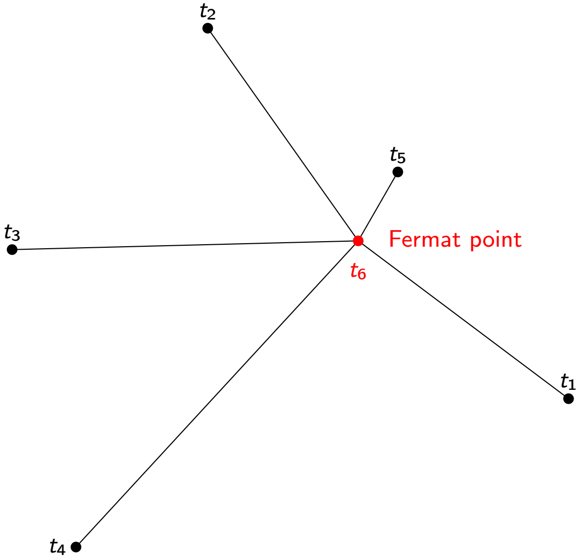
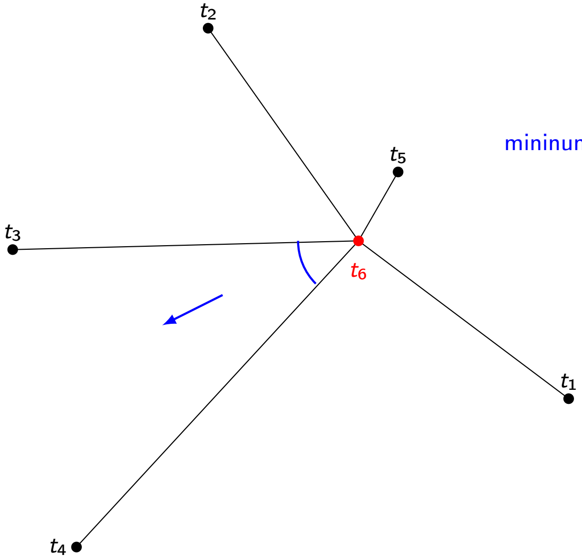
Heuristic - Example 1

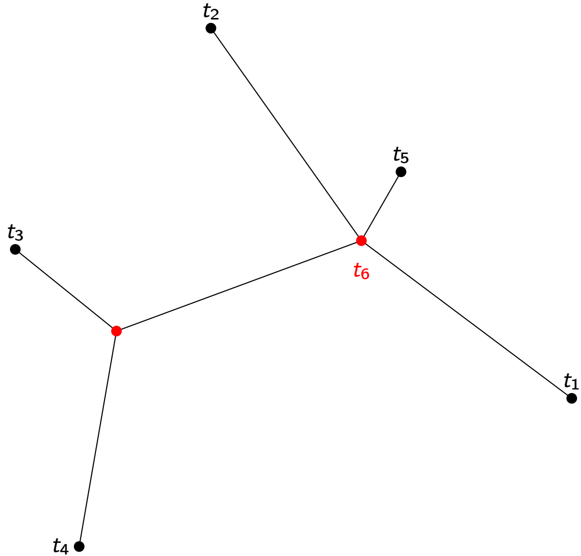


Heuristic - Example 1

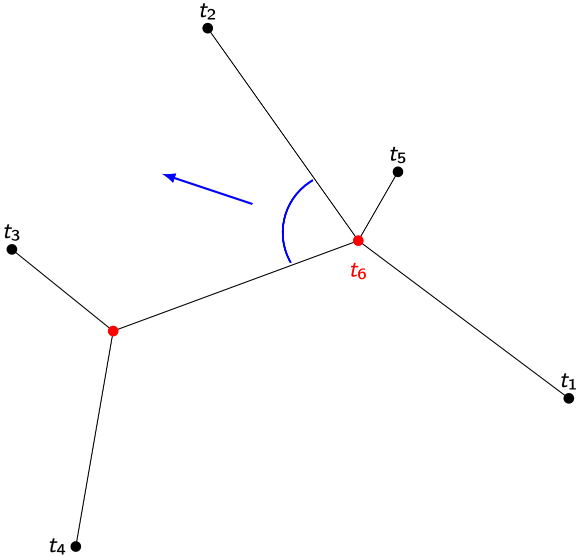


mininum angle?

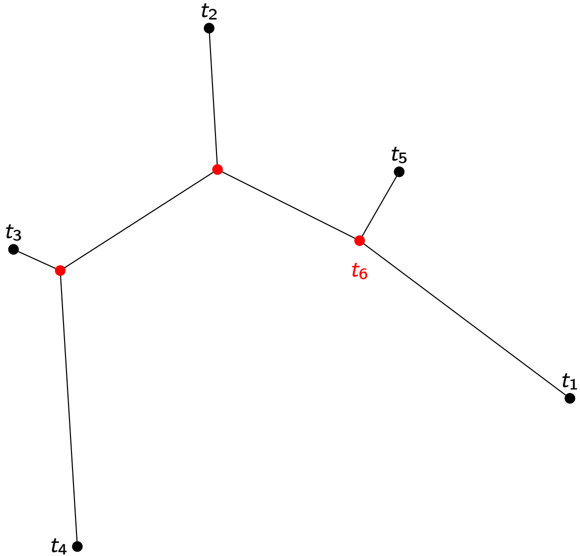
Heuristic - Example 1



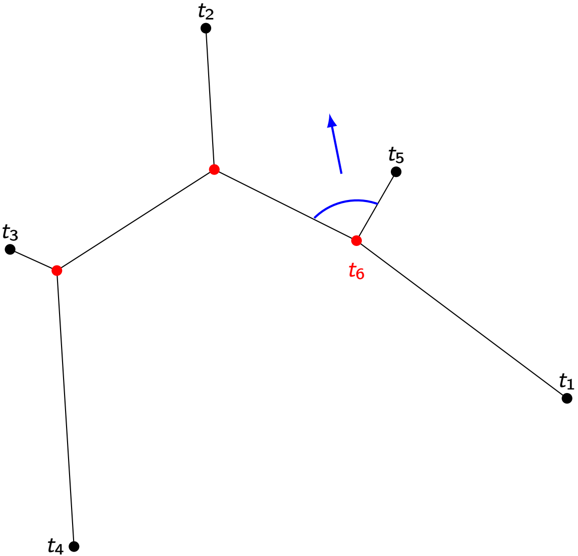
Heuristic - Example 1



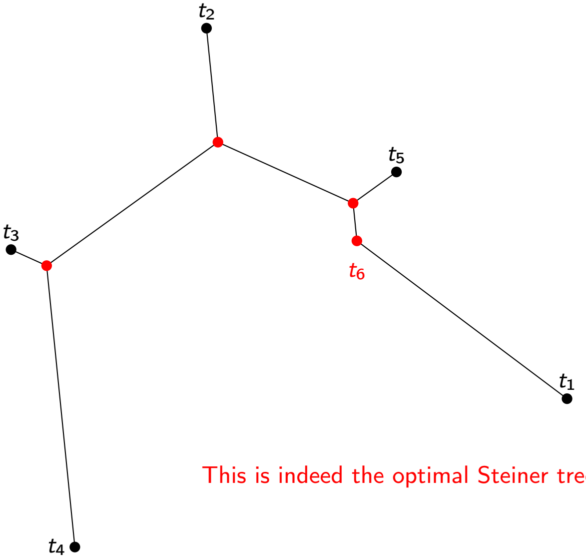
Heuristic - Example 1



Heuristic - Example 1

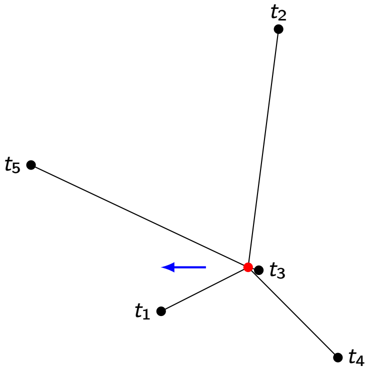


Heuristic - Example 1

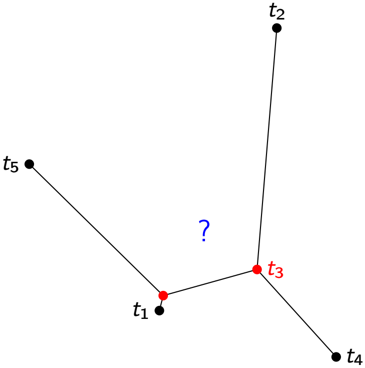


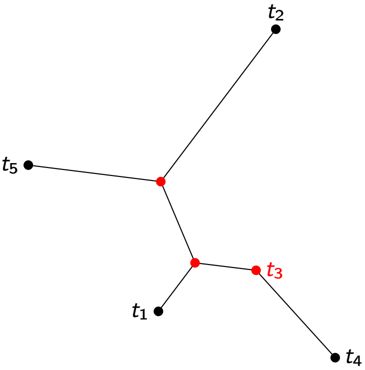
This is indeed the optimal Steiner tree.

Heuristic - Example 2



Heuristic - Example 2





From the topology in the left, no choice for the new Steiner point location will result in the optimal Steiner tree (depicted in the right).

Heuristics

Two new heuristics for the Euclidean Steiner Tree Problem in $\mathbb{R}^n$ [7]:

Heur1 : always choose minimum angle

Heur2 : tests all angles and pick the smallest resulting tree

Heuristics - Computational complexity

Let $D = \max_{i,j=1,\ldots,p} \| a^i - a^j \|$.

Consider a fixed topology and let E be the set of edges of this tree. An interior-point algorithm* can compute an ε-optimal solution in

$$O\left(\sqrt{|E|}\left(\log\left(\frac{D}{\varepsilon}\right) + \log|E|\right)\right)$$

iterations, where $|E|$ is the number of edges in the tree.

Using $D = 1$ and $\varepsilon = 10^{-16}$,

$$\sqrt{|E|}\left(log_{10}\left(\frac{D}{\varepsilon}\right) + log_{10}|E|\right) \leq \sqrt{2p-3}\left(16 + log_{10}(2p-3)\right).$$

Therefore, the computational complexity has order of

$$\sqrt{p} + \sqrt{p}\,\log(p)\,.$$

*G. Xue, Y. Ye. *An efficient algorithm for minimizing a sum of Euclidean norms with applications.* SIAM Journal on Optimization 4(7):1017–1036, 1997.

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Heuristic 1 ($D = 1$ and $\varepsilon = 10^{-16}$):

The search for the minimum angle is most costly in the first iteration, when there are p edges connecting every terminal to the Fermat-Weber point.

There are $p\,(p-1)/2 \;=\; O(p^2)$ pair of edges to have their angles computed. Therefore, at each iteration, the quantity of operations has order of

$$p^2 + \sqrt{p} + \sqrt{p}\,\log(p)\,.$$

Then the total of computational operations performed by Heuristic 1, after $O(p)$ iterations, has order of

$$p^3 + p\,\sqrt{p} + p\,\sqrt{p}\,\log(p)\,.$$

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Heuristic 2 ($D = 1$ and $\varepsilon = 10^{-16}$):

The model for a fixed topology has to be solved for each pair of consecutive edges. Then each iteration of Heuristic 2 has order of

$$p^2(\sqrt{p} + \sqrt{p}\,\log(p))\,.$$

As Heuristic 2 performs $p - 3$ iterations, the total of computational operations performed by Heuristic 2 has order of

$$p^3(\sqrt{p} + \sqrt{p}\,\log(p))\,.$$

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Numerical experiments

Platonic solids:

Instance	p	n	Exact	Heur1	Heur2	t1(s)	t2(s)
Tetrahedron	4	3	2.439	2.439	2.439	0.1	0.3
Octahedron	6	3	2.868	2.868	2.868	0.3	2
Cube	8	3	6.196	6.196	6.196	0.5	4.5
Icosahedron	12	3	18.553	18.553	18.664	0.9	18
Dodecahedron	20	3	22.911*	23.125	23.003	2	76

* Best solution found after 30 days of execution of Smith's algorithm [6].

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Hypercubes:

p	n	Conjecture*	Heur1	Heur2	t1(s)	t2(s)
16	4	13.124	13.124	13.124	1.4	38
32	5	26.981	26.981	26.981	5.5	247
64	6	54.694	54.694	54.694	24	1810
128	7	110.1192	110.286	110.2189	147	15000
256	8	220.9704	221.21	221.17	1060	150000

* Conjecture [6]: optimal solution value is $\left(2^{n-1} - 1\right)\sqrt{3} + 1$.

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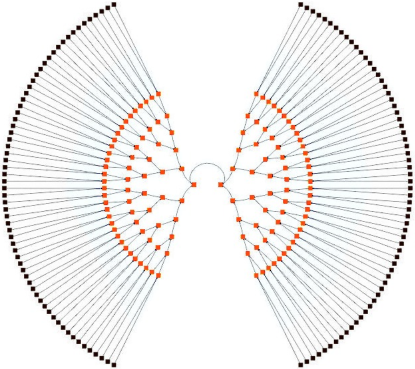
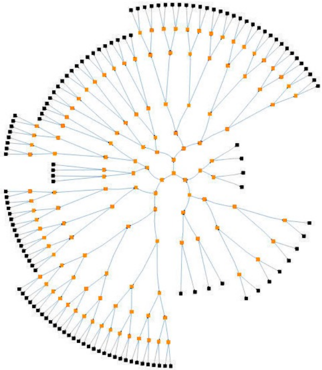
Hypercube in $\mathbb{R}^7$



Conjecture



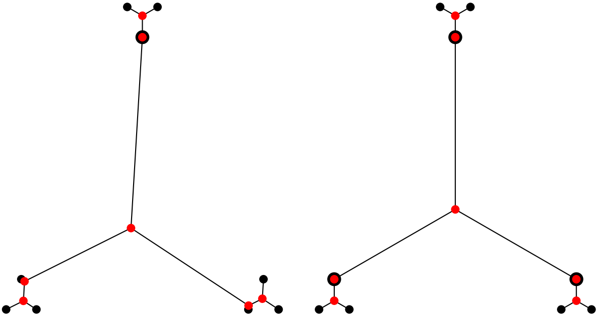
Heur2

The graphs were drawn using the tool availabe on the website https://www.yworks.com/yed-live/

Numerical experiments

Instance*	p	n	Exact	Heur1	Heur2	t1(s)	t2(s)
3-triangles	9	2	20.526	20.626	20.526	0.3	4
Ladder	10	2	8.3451	8.4641	8.3451	0.3	4.8

3-triangles:



Heur1 Heur2 (opt. sol.)

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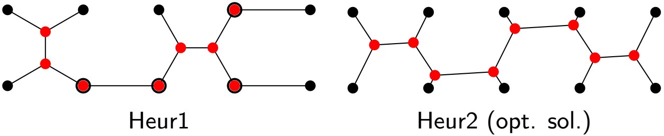
Numerical experiments

Instance*	p	n	Exact	Heur1	Heur2	t1(s)	t2(s)
3-triangles	9	2	20.526	20.626	20.526	0.3	4
Ladder	10	2	8.3451	8.4641	8.3451	0.3	4.8

Ladder:



* Instances from D.R. Dreyer and M.L. Overton. *Two heuristics for the Euclidean Steiner tree problem.* Journal of Global Optimization 13:95–106, 1998.

Numerical experiments

p	n	Heur2	tie	ILS	avg ρ	std
8	3	242	417	341	0.95074	0.022992
9	3	233	344	423	0.95124	0.021056
10	3	187	289	524	0.95260	0.021165
11	3	167	247	586	0.95427	0.021401
10	4	147	240	613	0.93231	0.020701
10	5	151	231	618	0.91502	0.019970

Instances from V. do Forte, F.M.T. Montenegro, J.A.M. Brito and N. Maculan. *Iterated local search algorithms for the Euclidean Steiner tree problem in n dimensions.* International Transactions in Operational Research (ITOR) 23(6):1185–1199, 2016.

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Random instances:

		Heur1			Heur2		
p	n	opt	gap	time	opt	gap	time
5	2	98%	0.01%	0.07	100%	0%	0.48
5	3	96%	0.03%	0.06	99%	0.002%	0.38
5	4	97%	0.005%	0.06	99%	0.004%	0.36
5	5	98%	0.01%	0.06	100%	0%	0.38
10	2	66%	0.7%	0.3	74%	0.5%	5
10	3	53%	0.6%	0.3	52%	0.5%	5
10	4	56%	0.5%	0.3	48%	0.5%	4.3
10	5	59%	0.2%	0.3	55%	0.3%	4.4

Numerical experiments

Random instances:

		Heur1		Heur2	
p	n	gap	time	gap	time
15	2	2%	0.6	1%	14
50	2	8%	9	2%	459
100	2	14%	60	3%	4530

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[3] M.H.C. Fampa and N. Maculan. Using a conic formulation for finding Steiner minimal trees. Numerical Algorithms, 35(4):315-330, 2004.

[4] H. Ouzia and N. Maculan. Mixed integer nonlinear optimization models for the Euclidean Steiner tree problem in $\mathbb{R}^d$, Journal of Global Optimization (JOGO), 83(1): 119-138, 2022.

[5] R.V. Pinto, H. Ouzia, N. Maculan. A new second-order conic optimization model for the Euclidean Steiner tree problem in $\mathbb{R}^d$. International Transactions in Operational Research, v. 1, p. 1, 2023.

[6] W. D. Smith. How to find Steiner minimal trees in Euclidean d-space. Algorithmica, 7:137–177, 1992.

[7] R.V. Pinto, N. Maculan. *A new heuristic for the Euclidean Steiner Tree Problem in* $\mathbb{R}^n$. TOP, v. 31, p. 391-413, 2023.

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Remembering my professors and co-authors who left us:

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Claude Berge

Egon Balas

Jacques Ferland

Francesco Maffioli

Clóvis Gonzaga

Michel Minoux

Merci beaucoup
Thank you
Muchas gracias
Muito obrigado
Hartelijk dank

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